TEMPERATURE DISTRIBUTION AT A THERMALLY INSULATED CRACK IN A PLATE AT VARIOUS BOUNDARY CONDITIONS

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Integral characteristics are defined for the temperature field of a plate with an open and thermally insulated crack and with boundary conditions of the first, the second, or the third kind at the lateral surfaces.

For the solution of thermoelasticity problems for thin plates or shells it is necessary to know the integral characteristics of the temperature field: the mean temperature T_1 and the temperature moment T_2, defined according to [1] as

$$T_1 = \int_0^1 t d\gamma, \quad T_2 = 12 \int_0^1 \left(\gamma - \frac{1}{2} \right) t d\gamma.$$

In view of this, it becomes worthwhile to solve the heat conduction problem so that both characteristics are immediately determined. In this case one may approximate the temperature distribution in the plate by the following expression [1]:

$$t = \left[1 - \frac{p^2}{12} (1 - 6\gamma + 6\gamma^2) \right] T_1 + \left[\gamma - \frac{1}{2} + \frac{p^2}{2} \left(\frac{1}{60} - \frac{\gamma}{5} + \frac{\gamma^2}{2} - \frac{\gamma^3}{3} \right) \right] T_2.$$

Inserting this value for t into the boundary conditions at the surfaces

$$a_1 \frac{\partial t}{\partial \gamma} + b_1 t = \Psi_1 \quad \text{for} \quad \gamma = 1;$$

$$-a_2 \frac{\partial t}{\partial \gamma} + b_2 t = \Psi_2 \quad \text{for} \quad \gamma = 0,$$

yields a system of differential equations

$$\sum_{j=1}^{2} L_{ij} (p^2) T_j = \Psi_i, \quad i = 1, 2, \tag{1}$$

where

$$L_{11} (p^2) = b_1 - \frac{p^2}{2} \left(a_1 + \frac{b_1}{6} \right); \quad L_{12} (p^2) = a_1 + \frac{b_1}{2} - \frac{p^2}{10} \left(a_1 + \frac{b_1}{12} \right);$$

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$$L_{21}(p^2) = b_2 - \frac{p^2}{2}\left(a_2 + \frac{b_2}{6}\right);$$

$$L_{22}(p^2) = -a_2 - \frac{b_2}{2} + \frac{p^2}{10}\left(a_2 + \frac{b_2}{12}\right),$$

and Ψ_i are known functions.

Assuming that a crack along y = 0, $|x| \le 1$ is thermally insulated, we obtain the following constraints on functions T_1 and T_2:

$$\frac{\partial T_j^+}{\partial y} = \frac{\partial T_j^-}{\partial y} = 0, j = 1, 2, \ |x| \leqslant 1.$$

Let us express the temperature t as a sum

$$t = t_0 + \vartheta,$$

where the first term represents the temperature field of a plate without cracks and the second term represents the temperature field perturbation due to a crack. The integral characteristics T_1 and T_2 will be expressed analogously as:

$$T_j = T_{j0} + \Theta_j, \ j = 1,2.$$

The quantities Θ_j will be found from the system of homogeneous differential equations

$$\sum_{j=1}^{2} L_{ij}(p^2)\Theta_j = 0, \ i = 1,2 \tag{2}$$

with the boundary conditions along the crack line y = 0

$$\frac{\partial \Theta_j^+}{\partial y} = \frac{\partial \Theta_j^-}{\partial y} = -f_j(x), \ f_j(x) = \frac{\partial T_{j0}^+}{\partial y}, \ j = 1,2, \ |x| \leqslant 1 \tag{3}$$

and with the requirement that both Θ_j vanish at infinity.

We introduce two new functions Φ and Φ_1:

$$\Theta_1 = L_{22}(p^2)\Phi - L_{12}(p^2)\Phi_1, \ \Theta_2 = L_{11}(p^2)\Phi_1 - L_{21}(p^2)\Phi. \tag{4}$$

Inserting (4) into system (2) yields two identities from which Φ and Φ_1 can be determined:

$$(ap^4 - bp^2 + c)\Phi = 0, \ (ap^4 - bp^2 + c)\Phi_1 = 0,$$

with

$$a = \frac{a_1 a_2}{10} + \frac{1}{80}(a_1 b_2 + a_2 b_1) + \frac{b_1 b_2}{720};$$

$$b = a_1 a_2 + \frac{13}{30}(a_1 b_2 + a_2 b_1) + \frac{b_1 b_2}{10};$$

$$c = a_1 b_2 + a_2 b_1 + b_1 b_2.$$

Consequently, $\Phi_1 = k\Phi$ with k = const. Letting k = 0, we have

$$\Theta_1 = L_{22}(p^2)\Phi, \ \Theta_2 = -L_{21}(p^2)\Phi. \tag{5}$$

With the aid of (5), we represent functions Θ_1 and Θ_2 in the form of Fourier integrals

$$\Theta_1^\pm = \pm \int_{-\infty}^{\infty} \sum_{i=1}^{2} L_{22}(\mu_i^2) N_i(s) \exp\left(-|y|\sqrt{s^2 + \varkappa_i^2} - isx\right) ds,$$
$$\Theta_2^\pm = \pm \int_{-\infty}^{\infty} \sum_{i=1}^{2} L_{21}(\mu_i^2) N_i(s) \exp\left(-|y|\sqrt{s^2 + \varkappa_i^2} - isx\right) ds, \tag{6}$$

where

$$\mu_1^2 = \frac{b - \sqrt{b^2 - 4ac}}{2a}, \ \mu_2^2 = \frac{b + \sqrt{b^2 - 4ac}}{2a}, \ \varkappa_i = \mu_i \frac{l}{h}.$$

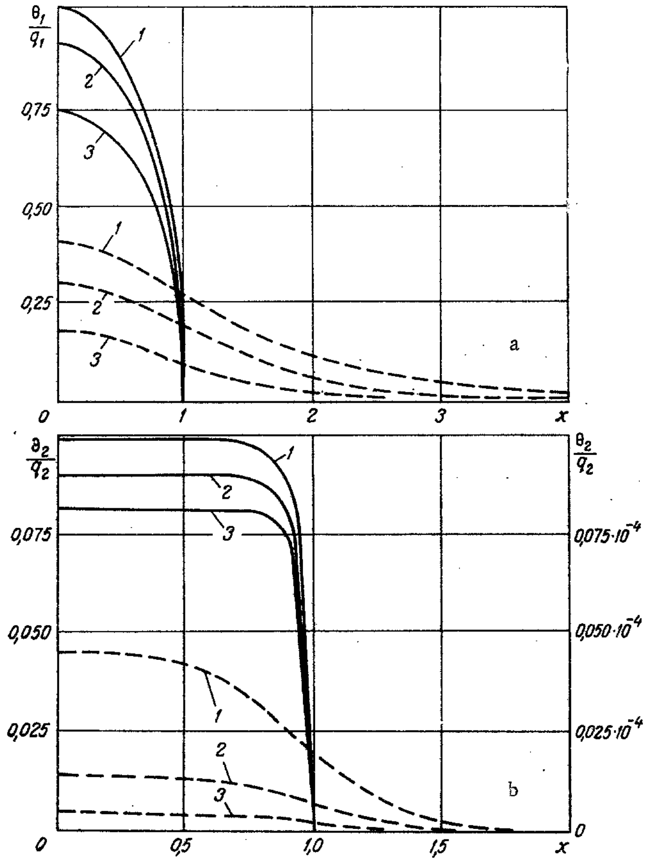

Fig. 1. Θ_1 (a) and Θ_2 (b) versus coordinates x and y. Solid curves, for y = 0; dashed curves, for y = 1. In Fig. 1b the scale for dashed curves is shown to the right.

We denote

$$\varphi_j(x) = \frac{1}{2}\left(\Theta_j^+ - \Theta_j^-\right) = \Theta_j^+ = -\Theta_j^-,\ j = 1{,}2. \tag{7}$$

Applying the Fourier transformation to these equalities and considering relation (6), we then solve the resulting system of equations and find

$$N_i(s) = \frac{1}{2\pi L\left(\mu_1^2,\ \mu_2^2\right)} \int_{-1}^{1} E_i(\xi)\exp(is\xi)\,d\xi.$$

where

$$E_1(\xi) = L_{21}\left(\mu_2^2\right)\varphi_1(\xi) + L_{22}\left(\mu_2^2\right)\varphi_2(\xi),$$
$$E_2(\xi) = -L_{21}\left(\mu_1^2\right)\varphi_1(\xi) - L_{22}\left(\mu_1^2\right)\varphi_2(\xi),$$
$$L\left(\mu_1^2,\ \mu_2^2\right) = L_{22}\left(\mu_1^2\right)L_{21}\left(\mu_2^2\right) - L_{22}\left(\mu_2^2\right)L_{21}\left(\mu_1^2\right).$$

Having satisfied the boundary conditions (3), we obtain for the unknown functions $E_1(\xi)$ and $E_2(\xi)$ two integral equations

$$\frac{\varkappa_i}{\pi}\int_{-1}^{1} E_i(\xi)\frac{K_1[\varkappa_i(\xi - x)]}{\xi - x}\,d\xi = -F_i(x),\ i = 1{,}2,\ |x| \leqslant 1, \tag{8}$$

where

$$F_1(x) = L_{21}\left(\mu_2^2\right)f_1(x) + L_{22}\left(\mu_2^2\right)f_2(x),$$
$$F_2(x) = -L_{21}\left(\mu_1^2\right)f_1(x) - L_{22}\left(\mu_1^2\right)f_2(x).$$

Solving these equations yields Θ_1 and Θ_2 at y = 0 according to formula (7), and at $y \neq 0$

$$\Theta_1 = \frac{y}{\pi L(\mu_1^2, \mu_2^2)} \sum_{i=1}^{2} \varkappa_i L_{22}(\mu_i^2) \int_{-1}^{1} E_i(\xi) \frac{K_1[\varkappa_i \sqrt{(\xi - x)^2 + y^2}]}{\sqrt{(\xi - x)^2 + y^2}} d\xi,$$

$$\Theta_2 = -\frac{y}{\pi L(\mu_1^2, \mu_2^2)} \sum_{i=1}^{2} \varkappa_i L_{21}(\mu_i^2) \int_{-1}^{1} E_i(\xi) \frac{K_1[\varkappa_i \sqrt{(\xi - x)^2 + y^2}]}{\sqrt{(\xi - x)^2 + y^2}} d\xi. \tag{9}$$

Equations (8) are structurally analogous to the integral equations of heat conduction problems for a plate with a crack and with symmetrical heat transfer at the lateral surfaces [3], or for a layer with an open crack [4]. In those references is shown an approximate solution to Eq. (8). With $F_i(x) = Q_i = \text{const.}$, for instance, the following closed approximate solution is found in [4]:

$$E_i(x) = \frac{Q_i}{\varkappa_i} \arccos \frac{\operatorname{ch} \varkappa_i x}{\operatorname{ch} \varkappa_i}. \tag{10}$$

For illustration, we will consider the case where boundary conditions of the third kind are specified at both plate surfaces and the heat transfer coefficients are the same ($a_1 = a_2 = 1$, $b_1 = b_2 = \varepsilon$). Then system (2) breaks down into two independent equations:

$$L_{21}(p^2)\Theta_1 = 0, \; L_{22}(p^2)\Theta_2 = 0.$$

Roots μ_1^2 and μ_2^2 are determined from the characteristic equations

$$L_{21}(\mu_1^2) = 0, \quad L_{22}(\mu_2^2) = 0.$$

Functions Θ_1 and Θ_2 at $y \neq 0$ will be found by the formula:

$$\Theta_i = \frac{y\varkappa_i}{\pi} \int_{-1}^{1} \varphi_i(\xi) \frac{K_1[\varkappa_i \sqrt{(\xi - x)^2 + y^2}]}{\sqrt{(\xi - x)^2 + y^2}} d\xi, \; i = 1,2. \tag{11}$$

For determining $\varphi_1(x)$ and $\varphi_2(x)$ we have two integral equations:

$$\frac{\varkappa_i}{\pi} \int_{-1}^{1} \varphi_i(\xi) \frac{K_1[\varkappa_i(\xi - x)]}{\xi - x} d\xi = -f_i(x), \; i = 1,2, \; |x| \leqslant 1.$$

Graphs of Θ_1/q_1 and Θ_2/q_2 have been plotted according to formula (11) and are shown in Fig. 1a,b as functions of x for $f_i(x) = q_i = \text{const.}$ and various values of ε, y with $l/h = 1$ ($\varepsilon = 0$, $\varkappa_1 = 0$, $\varkappa_2 = 10$; $\varepsilon = 0.26$, $\varkappa_1 = 0.5$, $\varkappa_2 = 11.06$; $\varepsilon = 0.55$, $\varkappa_1 = 1$, $\varkappa_2 = 12.17$). Solution (10) was used for the calculations. These graphs indicate that the integral characteristics of the temperature field at a crack and, therefore, the temperature field itself are local and attenuate faster with increasing ε.

NOTATION

t	is the temperature;
h	is the plate thickness;
$2l$	is the plate length;
x, y	are the dimensional rectangular coordinates, referred to l;
γ	is the thickness coordinate, reffered to h;
$p^2 = h^2/l^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$;	
$\varepsilon = \alpha h$;	
α	is the heat transfer coefficient;
$K_1(x)$	is the MacDonald function of the first order;
$T^{\pm} = \lim_{y \to \pm 0} T$.	

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